WESTERN SYDNEY
UNIVERSITY

Module 3

Matrix Operations

Matrix Addition and Subtraction

For $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$ and

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & A_{13} + B_{13} \\ A_{21} + B_{21} & A_{22} + B_{22} & A_{23} + B_{23} \end{bmatrix}$$

$$A - B = \begin{bmatrix} A_{11} - B_{11} & A_{12} - B_{12} & A_{13} - B_{13} \\ A_{21} - B_{21} & A_{22} - B_{22} & A_{23} - B_{23} \end{bmatrix}$$

Same as element-wise addition and subtraction!

Adding or Subtracting a Scalar

For c a scalar and $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$

$$A + c = \begin{bmatrix} A_{11} + c & A_{12} + c & A_{13} + c \\ A_{21} + c & A_{22} + c & A_{23} + c \end{bmatrix}$$

$$A - c = \begin{bmatrix} A_{11} - c & A_{12} - c & A_{13} - c \\ A_{21} - c & A_{22} - c & A_{23} - c \end{bmatrix}$$

Matrix Multiplication

- MATLAB denotes matrix multiplication with an asterisk (*)
- In general, the product between two matrices is **not commutative**, i.e., $\boldsymbol{A} * \boldsymbol{B} \neq \boldsymbol{B} * \boldsymbol{A}$
- For matrix multiplication to work, the number of columns in left matrix must be same as number of rows in right matrix
  - Also known as the "inner dimensions"

$$\mathrm{C}_{m \times n} = A_{m \times p} \; B_{p \times n}$$

Showing non-commutative nature of matrices

```
>> A = randi(3,3)
A =
     3     3     1
     3     2     2
     1     1     3
>> B=randi(3,3)
B =
     3     3     1
     1     2     2
     3     3     3
>> AB = A*B
AB =
    15    18    12
    17    19    13
    13    14    12
```

```
>> BA = B*A
BA =
    19    16    12
    11     9    11
    21    18    18
>> AB == BA
ans =
     0     0     1
     0     0     0
     0     0     0
```

Example 1: Matrix multiplication

$$A = \begin{bmatrix} 1 & 2 & 6 \\ -2 & 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 \\ 7 & 5 \\ -2 & -4 \end{bmatrix}$$

Find the product

$$C = A \times B$$

Example 1 (cont.)

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} \Rightarrow c_{12} = \sum_{k=1}^{3} a_{1k} b_{2j}$$

$$A = \begin{bmatrix} 1 & 2 & 6 \\ -2 & 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 1 \\ 7 & 5 \\ -2 & -4 \end{bmatrix}$$

$$c_{12} = \begin{bmatrix} 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$$

$$= 1 \times 1 + 2 \times 5 + 6 \times (-4) = -13$$

Example 1 (cont.)

$$C = \begin{bmatrix} 1 \times 3 + 2 \times 7 + 6 \times (-2) & 1 \times 1 + 2 \times 5 + 6 \times -4 \\ -2 \times 3 + 4 \times 7 + 1 \times (-2) & -2 \times 1 + 4 \times 5 + 1 \times -4 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & -13 \\ 20 & 14 \end{bmatrix}$$

Example 2: Matrix multiplication

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

then the matrix that is obtained with the operation $A*B$ has dimensions 4×2 with the elements:

$$\begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}) & (A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}) \\ (A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}) & (A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32}) \\ (A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31}) & (A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32}) \\ (A_{41}B_{11} + A_{42}B_{21} + A_{43}B_{31}) & (A_{41}B_{12} + A_{42}B_{22} + A_{43}B_{32}) \end{bmatrix}$$

A numerical example:

$$\begin{bmatrix} 1 & 4 & 3 \\ 2 & 6 & 1 \\ 5 & 2 & 8 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} (1 \cdot 5 + 4 \cdot 1 + 3 \cdot 2) & (1 \cdot 4 + 4 \cdot 3 + 3 \cdot 6) \\ (2 \cdot 5 + 6 \cdot 1 + 1 \cdot 2) & (2 \cdot 4 + 6 \cdot 3 + 1 \cdot 6) \\ (5 \cdot 5 + 2 \cdot 1 + 8 \cdot 2) & (5 \cdot 4 + 2 \cdot 3 + 8 \cdot 6) \end{bmatrix} = \begin{bmatrix} 15 & 34 \\ 18 & 32 \\ 43 & 74 \end{bmatrix}$$

Matrix Division

- There is no matrix division!
- The only matrix operations that exist are:
  - Addition and subtraction
  - Multiplication
  - Scalar multiplication
  - Transposition

When performing matrix multiplication on two square matrices

- They must both have the same dimensions
- The result is a matrix of the same dimension
- In general, the product is not commutative, i.e., $A * B \neq B * A$

Matrix Multiplication

$$\mathrm{C}_{m\times n} = A_{m\times p}\, B_{p\times n}$$

The elements of matrix C can be found by

$$c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}$$

for each $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$

That is, *i*th row and *j*th column element of matrix **C** is found by multiplying the *i*th row of **A** and *j*th column of **B**.

Matrix Exponentiation

- The matrix operation $A \times A$ means A^2
- That is `A^2` means matrix **A** multiplied by matrix **A** (not element-wise multiplication)

Command Window

```
>> A
A =
     2     1     4
     4     1     8
     2    -1     3
>> A^2
ans =
    16    -1    28
    28    -3    48
     6    -2     9
>> 
```

Command Window

```
>> A*A
ans =
    16    -1    28
    28    -3    48
     6    -2     9
>> A.*A
ans =
     4     1    16
    16     1    64
     4     1     9
>> 
```

Same as A^2 (>> A*A)

not equal to A^2 (>> A.*A)

When performing matrix multiplication on two vectors:

- They must both be the same size
- One must be a row vector and the other a column vector
- If the row vector is on the left, the product is a scalar
- If the row vector is on the right, the product is a square matrix whose side is the same size as the vectors

Identity matrix

- Square matrix with ones on main diagonal and zeros elsewhere
- When do matrix multiplication on any array or vector with the identity matrix, array or vector is unchanged
  - True whether multiply with identity matrix on left or on right
- MATLAB command eye(n) makes an $n \times n$ identity matrix

Identity matrix

```
>> I=eye(3)
I =
     1     0     0
     0     1     0
     0     0     1
>> a*I
ans =
     4     5     7
>> I*a'
ans =
     4
     5
     7
>>
```

```
>> B=magic(3)
B =
     8     1     6
     3     5     7
     4     9     2
>> I*B
ans =
     8     1     6
     3     5     7
     4     9     2
>> B*I
ans =
     8     1     6
     3     5     7
     4     9     2
>>
```

Inverse of a Matrix

Matrix B is the *inverse* of matrix A if matrix product of A and B is the identity matrix I

- Both matrices must be square and have the same dimensions
- Multiplication can be from either side, i.e.,

$$BA = AB = I$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 8 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 5.5 & -3.5 & 2 \\ 2 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5.5 & -3.5 & 2 \\ 2 & -1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 8 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In math, inverse of a matrix A is written as A^{-1}

In MATLAB, get inverse with `A^-1` or `inv(A)`

```
>> A=[2 1 4;4 1 8;2 -1 3]
A =
     2     1     4
     4     1     8
     2    -1     3
>> B=inv(A)
B =
    5.5000   -3.5000    2.0000
    2.0000   -1.0000         0
   -3.0000    2.0000   -1.0000
>> C=A^-1
C =
    5.5000   -3.5000    2.0000
    2.0000   -1.0000         0
   -3.0000    2.0000   -1.0000
>>
```

```
>> A*inv(A)
ans =
     1     0     0
     0     1     0
     0     0     1
>> A^-1*A
ans =
     1     0     0
     0     1     0
     0     0     1
>>
```

Determinants

A *determinant* is a function associated with square matrices

- In math, determinant of A is written as *det(A)* or $|A|$
- In MATLAB, `det(A)` gives determinant of A
- A matrix has an inverse only if it is square and its determinant is not zero

Determinants

Command Window

```
>> A=[2 1 4;4 1 8;2 -1 3]
A =
     2     1     4
     4     1     8
     2    -1     3
>> det(A)
ans =
     2
>> inv(A)
ans =
    5.5000   -3.5000    2.0000
    2.0000   -1.0000         0
   -3.0000    2.0000   -1.0000
>>
```

Command Window

```
>> B=[2 1 4;4 1 8;4 -2 8]
B =
     2     1     4
     4     1     8
     4    -2     8
>> det(B)
ans =
     0
>> inv(B)
Warning: Matrix is singular to
working precision.
ans =
   Inf   Inf   Inf
   Inf   Inf   Inf
   Inf   Inf   Inf
>>
```

Matrix functions

- `repmat` – repeat columns and rows
- `dot(v,w)` – dot (inner product); v, w both vectors of same size but any dimension
- `cross(v,w)` – cross product; v, w must both have three elements but any dimension
- `det(A)` – determinant of square matrix A
- `inv(A)` – inverse of square matrix A

More matrix functions

- `repmat` – repeat columns and rows

```
Command Window
>> a
a =
     1     2     3
>> repmat(a,[3 1])
ans =
     1     2     3
     1     2     3
     1     2     3
>> repmat(a,[3 2])
ans =
     1     2     3     1     2     3
     1     2     3     1     2     3
     1     2     3     1     2     3
>>
```